6 More on the 1D Wave Equation on the Line

We discuss here a number of basic points about the Cauchy problem

$$\begin{aligned} &\tfrac{\partial^2 u}{\partial t^2} = c^2 \tfrac{\partial^2 u}{\partial x^2}, \quad |x| < \infty, t > 0 \\ &u(x,0) = f(x), \tfrac{\partial u}{\partial t}(x,0) = g(x), \quad |x| < \infty \end{aligned} \tag{1}$$

6.1 Additional important concepts

Remark: domain of dependence, region of influence
Through every point in the domain $\mathbb{R} \times \mathbb{R}^+$ there are two characteristics, and for our constant-coefficient wave equation, these are straight lines with one having positive slope, the other having negative slope. Thus, they form a triangle with the x-axis, the **characteristic triangle**, and as d'Alembert's formula shows, it is the initial data at the base of the triangle that is important (see Figure 1). So the solution to (1) is composed of two components, $u^{[1]}$ and $u^{[2]}$, namely $u^{[1]}(x,t) = \{f(x+ct) + f(x-ct)\}/2$, the contribution from the initial displacement, and $u^{[2]}(x,t) = [\int_{x-ct}^{x+ct} g(y)dy]/2c$, the contribution from the initial velocity of the string. Therefore, given the (homogeneous) wave equation, for each point (x_0, t_0) in its domain, there is a **domain of dependence**, namely the (**closed**) interval $[x_0 - ct_0, x_0 + ct_0]$. Put another way, $u^{[1]}$ depends on averaging the initial displacement at endpoints $(x_0 \pm ct_0, 0)$, and $u^{[2]}$ depends on the average of the initial velocity $g(\cdot)$ over the interval $(x_0 - ct_0, x_0 + ct_0)$. This leads to another definition, that of the **region of influence**. That is, given point $(x,t) = (x_0, 0)$, its **region of influence** is the part of the domain between the characteristics $x - ct = x_0$ and $x + ct = x_0$ (see Figure 2) because any point in that region has a characteristic triangle whose base interval contains x_0. Thus, the solution at a point in the region of influence of x_0 is influenced by any non-zero data imposed at $(x_0, 0)$.

Remark: well-posedness
Given sufficient smoothness, as presented in the Theorem in Section 8, we can also say that the IVP (1) is "structurally stable" in the following sense: For any interval of time $[0,T]$, with $T > 0$, and any limiting error $\varepsilon > 0$, a $\delta > 0$ can be found such that two solutions to (1), u_1 and u_2, will differ by less than ε, i.e.
$|u_1(x,t) - u_2(x,t)| < \varepsilon$, for all $x \in \mathbb{R}, t \in [0,T]$,

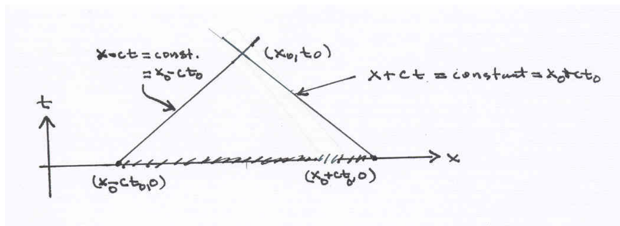

Figure 1: The characteristic triangle for point (x_0, t_0)

provided $u_1(x,0) = f_1(x), \frac{\partial u_1}{\partial t}(x,0) = g_1(x), u_2(x,0) = f_2(x), \frac{\partial u_2}{\partial t}(x,0) = g_2(x)$ differ by less than δ:
$|f_1(x) - f_2(x)| < \delta$, $|g_1(x) - g_2(x)| < \delta$ for all $x \in \mathbb{R}$. Hence, small changes in initial data lead to small changes in the solution u.

The proof of this is straightforward. By d'Alembert's formula, with u_j being the solution of (1) with data $\{f_j, g_j\}$, $j = 1, 2$, then

$$|u_1(x,t) - u_2(x,t)| =$$

$$|\frac{f_1(x+ct) + f_1(x-ct)}{2} - \frac{f_2(x+ct) + f_2(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_1(y)dy - \frac{1}{2c}\int_{x-ct}^{x+ct} g_2(y)dy| \leq$$

$$\frac{|f_1(x+ct) - f_2(x+ct)|}{2} + \frac{|f_1(x-ct) - f_2(x-ct)|}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} |g_1(y) - g_2(y)|dy \leq$$

$$\frac{\delta}{2} + \frac{\delta}{2} + \frac{1}{2c}\delta(2ct) = \delta + \delta t \leq \delta(1+T),$$

which proves the statement if we put $\delta < \varepsilon/(1+T)$. Note that *uniqueness* of solutions to (1) is an automatic consequence of this calculation; if we have two solutions, u_1 and u_2, to (1) with data $\{f, g\}$, then $|u_1(x,t) - u_2(x,t)| \leq 0$ from the above inequality ($f_1 = f_2 = f, g_1 = g_2 = g$), which implies $u_1 \equiv u_2$ for $(x,t) \in \mathbb{R} \times [0,T]$ for all $T > 0$.

This stability property of (1) is very desirable because it means if we make small errors in determining the initial velocity and/or displacement, we can expect to have just small errors in the solution. A problem is said to be **well-posed** if the following three conditions are met:

1. the problem has a solution
2. the solution is unique

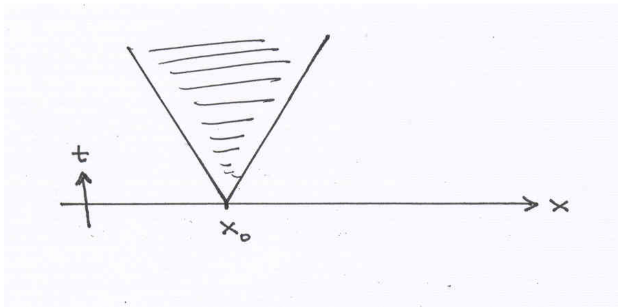

Figure 2: The shaded area illustrates the region of influence for point $(x_0, 0)$

3. the solution is stable in the above sense

Hence, for smooth data, the Cauchy problem (1) is well-posed. In these Notes we will generally only discuss well-posed problems, but keep in mind that there are important physical problems that are **not** well-posed.

Note also, that this definition of well-posedness is a bit vague, since the notion of solution is not specified, nor is the notion of a solution being stable. If the data in the problem do not have the smoothness specified in the theorem Section 5, page 5, we can not automatically take derivatives of the d'Alembert formula and substitute it back into the equation, though the formula still has value. This leads us to consider weakening the notion of solution. A form of this is discussed in section 6.3.1. For a notion of a solution being stable, we will stick with how be defined it above.

Remark: conservation of energy

As indicated in the examples of the last section, there is a *conservation of energy principle* going on with our 1D wave equation. To demonstrate this, recall from your physics course that kinetic energy of an object of mass m and velocity v is $\frac{1}{2}mv^2$. In the context of our infinite string, the kinetic energy is the sum of all such contributions, so, in our scaled variables, write

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{\partial u}{\partial t}\right)^2 dx.$$

Also, in our scaled variables, the system's potential energy is given by

$$PE = \frac{c^2}{2}\int_{-\infty}^{\infty} \left(\frac{\partial u}{\partial x}\right)^2 dx$$

So the total energy, $E = KE + PE$, is

$$E = E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \{(u_t)^2 + c^2(u_x)^2\} dx. \tag{2}$$

This assumes $u(x,t)$ is the (classical) solution to (1), and that u_t^2 and u_x^2 are integrable (so E is finite). Hence

$$\int_{-\infty}^{\infty} (u_x)^2 \, dx < 0 \text{ and } \int_{-\infty}^{\infty} (u_t)^2 \, dx < 0 \text{ implies } u_x, u_t \to 0 \text{ as } |x| \to \infty \, .$$

Now differentiate E:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \{u_t u_{tt} + c^2 u_x u_{xt}\} dx.$$

But

$$\int_{-\infty}^{\infty} u_x u_{xt} dx = u_x u_t|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_t u_{xx} dx = -\int_{-\infty}^{\infty} u_t u_{xx} dx,$$

so

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t\{u_{tt} - c^2 u_{xx}\} dx = 0.$$

Therefore, E is independent of t, that is, $E =$ constant, for all t (so is given by the initial data at $t = 0$), and this is just a statement of the energy conservation principle.

Remark: uniqueness
We can also obtain uniqueness of solutions using the energy argument. Suppose there are two solutions, u_1 and u_2, to the Cauchy problem (1). Then $u := u_1 - u_2$ satisfies the wave equation, and $u(x,0) = u_t(x,0) \equiv 0$. Using the energy expression $E(t)$ in (2), we have $E(t) \geq 0$ from the definition, $dE(t)/dt = 0$ via the above calculation, and $E(0) = 0$ because of $u's$ initial conditions. Hence, for all $t \geq 0$, $E(t) = 0$, which implies $u_x^2 \equiv 0, u_t^2 \equiv 0$. So $u =$ constant, but because u is continuous for $t \geq 0$, and $u(x,0) = 0$ for $|x| < \infty$, then $u \equiv 0$ everywhere, which means $u_1 \equiv u_2$ everywhere.

Here are some extra practice problems:
Exercises

1. Solve
$$\begin{cases} u_{tt} - 4u_{xx} = 0 & |x| < \infty, t > 0 \\ u(x,0) = u_t(x,0) = 2\sin(3x) & |x| < \infty \end{cases}$$

2. Given $u_{tt} = 2u_{xx}$ for $|x| < \infty, t > 0$, $u(x,0) = H(x)$, $u_t(x,0) = H(-x)$, map out what the d'Alembert solution is in its domain. In particular, what is $u(x,t)$ at $(x,t) = (1/2, 1/\sqrt{2})$? What is its domain of dependence?

3. Determine the solution to the wave equation in the various regions of the domain for the problem
$$\begin{cases} u_{tt} - c^2u_{xx} = 0 & |x| < \infty, t > 0 \\ u(x,0) = 2H(x+1), u_t(x,0) = H(x-1) & |x| < \infty \end{cases}$$

6.2 Solution of the Cauchy problem with source terms

Consider the problem
$$\begin{cases} u_{tt} = c^2u_{xx} + F(x,t) & |x| < \infty, t > 0 \\ \\ u(x,0) = f(x), u_t(x,0) = g(x) & |x| < \infty \end{cases} \tag{3}$$

Theorem: If f and g are continuous, with continuous derivatives f', f'', g' on $\mathbb{R}$, and F is continuous on $\mathbb{R} \times \mathbb{R}^+$, and for each $t > 0$, F is integrable, then the solution to (4) is
$$u(x,t) = \frac{1}{2}\{f(x+ct)+f(x-ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy+\frac{1}{2c}\int\int_{\Delta(x,t)} F(y,\tau)dyd\tau \tag{4}$$
where $\Delta(x,t)$ is the characteristic triangle associated with point (x,t) at its apex, so
$$\int\int_{\Delta(x,t)} F(y,\tau)dyd\tau = \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(y,\tau)dyd\tau. \tag{5}$$
Remark: By linearity of the equation, $u = u_1 + u_2$, where u_1 is a solution to problem (4) without the forcing term F, hence contributing the first two terms of (5) via d'Alembert's formula, while u_2 is a solution to (4) with $f = g = 0$. So, for a problem with zero initial data, but with source term F, u_2 would be equal to the last double integral of (5).

There are a number of ways to obtain (5) as the solution to (4), but one of the simplest ways is to use Green's theorem from calculus. We will go through the argument in subsection 6.3.

Example 1:

$$\begin{cases} u_{tt} = u_{xx} + x & |x| < \infty, t > 0 \\ u(x,0) = \sin(x), u_t(x,0) = 0 & |x| < \infty \end{cases}$$

(5) gives $u(x,t) = \frac{1}{2}\{\sin(x+t) + \sin(x-t)\} + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} y dy d\tau$. First,

$$\begin{aligned} \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} y dy d\tau &= \int_0^t (\frac{y^2}{2}|_{x-t+\tau}^{x+t-\tau}) d\tau \\ &= \frac{1}{2}\int_0^t \{(x+t-\tau)^2 - (x-t+\tau)^2\} d\tau \\ &= 2\int_0^t x(t-\tau) d\tau = xt^2 \ , \end{aligned}$$

and

$$\tfrac{1}{2}\{sin(x+t) + \sin(x-t)\} = \tfrac{1}{2}\{\sin(x)\cos(t) + \cos(x)\sin(t)$$

$$+ \sin(x)\cos(t) - \cos(x)\sin(t)\} = \sin(x)\cos(t) \ .$$

Therefore, $u(x,t) = \sin(x)\cos(t) + \frac{xt^2}{2}$.

Example 2:

$$\begin{cases} u_{tt} = c^2 u_{xx} + \cos(x) & |x| < \infty, t > 0 \\ u(x,0) = 0, \ u_t(x,0) = 1 + x & |x| < \infty \end{cases}$$

Now

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} (1+y) dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \cos(y) dy \ d\tau \ .$$

Note,

$$\int_{x-ct}^{x+ct} (1+y) dy = 2ct + \frac{1}{2}\{(x+ct)^2 - (x-ct)^2\} = 2ct + 2cxt = 2ct(1+x).$$

Finally,

$\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \cos(y) dy d\tau = \int_0^t \{\sin(x+c(t-\tau)) - \sin(x-c(t-\tau))\} d\tau$

$= [\sin(x+ct) - \sin(x-ct)] \int_0^t \cos(c\tau)d\tau - [\cos(x+ct) + \cos(x-ct)] \int_0^t \sin(c\tau)d\tau$

$= \frac{2}{c}\cos(x)(1-\cos(ct))$.

Putting these together, we have
$u(x,t) = t(1+x) + \frac{1}{c^2}\cos(x)(1-\cos(ct))$.

Exercises

1. Solve
$$\begin{cases} u_{tt} = c^2 u_{xx} + xt & |x| < \infty, t > 0 \\ u(x,0) = 0, u_t(x,0) = 0 & |x| < \infty \end{cases}$$
(ans: $u(x,t) = xt^3/6$)

2. Solve
$$\begin{cases} u_{tt} = c^2 u_{xx} + e^{ax} & |x| < \infty, t > 0 \\ u(x,0) = 0, u_t(x,0) = 0 & |x| < \infty \end{cases}$$
(ans: $u(x,t) = e^{ax}\{cosh(act) - 1\}/(ac)^2$)

6.3 Appendix to the Section

6.3.1 Weak solution of the 1D wave equation

The solution to the problem (1), as expressed by d'Alembert's formula, is only a classical solution as expressed by the Theorem in Section 5, page 5, if the data $\{f, g\}$ is smooth enough. In illustrating the interpretation of d'Alembert's solution we used discontinuous initial displacement and/or initial velocity, and showed that singularities are propagated (with finite speed) by the characteristics. The characteristics that propagate the singularities break the domain into a number of regions where the solution exists in the classical sense in each region. But the solution is not smooth on the characteristics. I want to extend the notion of solution using a algebraic property.

Suppose we go back to the general solution of the wave equation, namely $u(x,t) = F(x-ct) + G(x+ct) = F(\xi) + G(\eta)$, but we no longer require F, G to be C^2 on domain $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Consider rectangle ABCD in

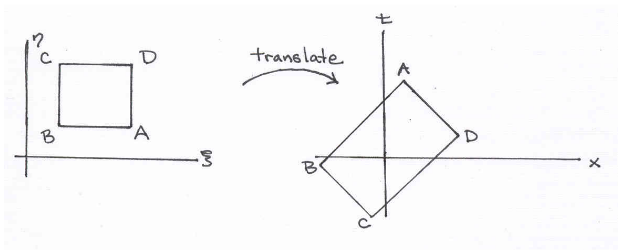

Figure 3: This figure is for illustrating the parallelogram rule

ξ, η plane (Figure 3). The sides of the rectangle are parallel to the axes, so $F(\xi)$ = constant on vertical lines, $G(\eta)$ = constant on horizontal lines. Hence, $F(A) = F(D)$, $F(B) = F(C)$, $G(A) = G(B)$, $G(C) = G(D)$, and so $u(\xi, \eta) = F(\xi) + G(\eta)$ implies

$$u(A) + u(C) = u(B) + u(D) \ , \tag{6}$$

so sums of values of u at opposite vertices are equal. After translating to the x, t plane (right graph in Figure 3) the parallelogram has sides parallel to the characteristic directions. View (6) as a **parallelogram rule**. Therefore, a **weak solution** of the wave equation is defined to be any function $u(x, t)$ satisfying (6) *for all* parallelograms in its domain with sides as segments of characteristics.

6.3.2 Proof of the theorem in section 6.2

We want to make use of Green's theorem

$$\int\int_{\Delta} (P_x - Q_t) dt dx = \int_{\partial\Delta} P dt + Q dx.$$

(This also allows you to see the use of line integral and use of this (calculus) theorem again. See Figure 4 for notation.)

Let $P = -c^2 u_x$ and $Q = -u_t$. Then $P_x = -c^2 u_{xx}$ and $-Q_t = u_{tt}$, so $P_x - Q_t = u_{tt} - c^2 u_{xx}$. Therefore,

$$\int\int_{\Delta} (P_x - Q_t) dt dx = \int\int_{\Delta} (u_{tt} - c^2 u_{xx}) dx dt = \int\int_{\Delta} F dx dt,$$

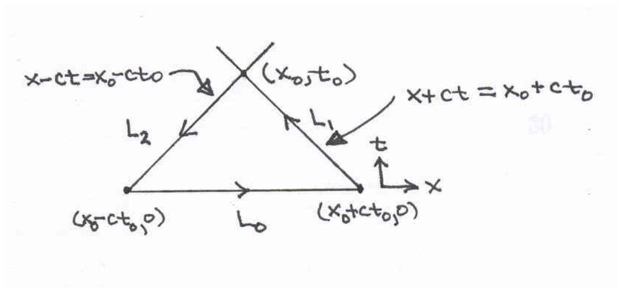

Figure 4: This figure gives the notation used in the proof of the Theorem

but the left-hand side is also equal to

$$\oint_{\partial\Delta} Pdt + Qdx = \int_{L_0+L_1+L_2} \{-c^2 u_x dt - u_t dx\}.$$

Now along L_0, $dt = 0$, so with $u_t(x, 0) = g(x)$, this implies

$$\int_{L_0} (Pdt + Qdx) = -\int_{x_0-ct_0}^{x_0+ct_0} g(y)dy.$$

On L_1, $x + ct = x_0 + ct_0$, so $dx + cdt = 0$, or $dx = -cdt$, so $-c^2u_xdt - u_tdx = cu_xdx - u_tdx = cu_xdx + cu_tdt = cdu$. Hence,

$$\int_{L_1} (Pdt + Qdx) = c\int_{L_1} du = cu|_{(x_0+ct_0,0)}^{(x_0,t_0)} = cu(x_0, t_0) - cf(x_0 + ct_0).$$

On L_2, $x - ct = x_0 - ct_0$, so $dx = cdt$, which gives

$$\int_{L_2} (Pdt + Qdx) = -c\int_{L_1} du = -cu|_{(x_0,t_0)}^{(x_0-ct_0,0)} = -cf(x_0 - ct_0) + cu(x_0, t_0).$$

Adding the three integrals together gives

$$\int\int_{\Delta} Fdxdt = -\int_{x_0-ct_0}^{x_0+ct_0} g(y)dy + 2cu(x_0, t_0) - cf(x_0 + ct_0) - cf(x_0 - ct_0).$$

Rearranging terms and dividing by $2c$ gives solution formula (4).